Comparsision of numerical integration

by: **Ariane-Tabea Schüller**

1 Preface

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Task

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The following paper contains a comparison of different numerical integration-methods, by Ariane-Tabea Schüller. I have compounded the rectangular - , Simpson - and trapezoid-rule with each other. This one can see by looking at the examples of the various functions used to show the differences, which were calculated and evaluated by myself.

History of integration and general terms

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Important inventions as the “Relativitätstheorie” by Albert Einstein took many years to be developed. Often it starts with a coincidence followed by an idea, which turns after many years of hard work into a completely elaborated draft. If one looks at mathematics there are not many differences. Therefore, events as the landing on the moon were calculated by easy and very old calculation-methods.
The roots of numerical integration one can already find in the antique times of Greek, where scholars tried to determine the contents of the level. In 1615 Kepler reintroduced and carried on this fundamental idea in his work “nova stereometria doliorum vinariorum”, whereupon the integral calculus of accurate level-determination was discovered. Today this kind of level-determination is used in many different ways, as in electrical engineering, dynamics, statics, and many more.

In general one must be able to construe an positve integral of a constant function as a level in order to solve integrals. This level is situated below the indegrand of the appointed function and limited to the interval [a,b]. If the stammfunction F of the integrand f can be constituted, the following formula applies:

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

If the stammfunction F of the indegrand f cannot be constitued, the integral can only be solved approximately. For example:

$$\int_a^b 2\,e^{3x^2}\,dx = 2\int_a^b e^{3x^2}\,dx$$

At this point of calculations various numerical integration methods turn up - quadrature formulas - which fairly define the value of the integral, e.g. Riemansche Ober- and Untersumme. Rectangle-, trapezoid-, and simpsonrule are similar in shape but because of simple geometrical considerations they result in more precise outcomes.

2 The rectangular-rules

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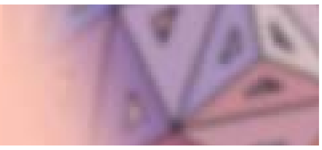
General terms about the rectangular-rues



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The definition of the Riemmansche integral for constant and differentiately functions, y = f(x), states that the integration interval [a,b] is subdivided by the points x_1, x_2, ..., x_{n-1} into n equal rectangle-levels, each of the length of $h = \frac{b-a}{n}$.

This approximation is also known as rectangle-rule. To determine the height of the rectangle-levels one has to use the left or the right function value of the subinterval. According to this, there are two different approximate values:

(1) left function value= Untersumme =lenght * left hight

$$\int_a^b f(x)\,dx \approx h\times(y_0+y_1+y_2+....+y_{n-1})$$

$$\bigwedge_{h\in\mathbb{N}} U_n \le O_n \qquad \lim_{h\to\infty}(O_n-U_n)=0$$

(2) right function value = Obersumme = lenght * right hight

$$\int_a^b f(x)\,dx \approx h\times(y_1+y_2+.....+y_n)$$

After the rectangle-levels, above a subinterval, are summed up this process is finished. The method is called rectangle-procedure, one of the simplest numerical integration procedures. For further calculations it does not matter if one uses the left or the right function value to determine the level, because a high number of n makes both sums converge to the same limiting value.

The rectangle-rule is originated form Ober-, and Untersumme and generally speaking it achieves more accurate results.

$$\int_a^b f(x)\,dx \approx h\times\sum_{i=0}^{n-1} f(a+i\times h) \tag{3}$$

3

Trapezoid-rule

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General terms - trapezoid-rule

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To obtain an essential improvement of the numerical integral value the rectangle-levels are exchanged for trapezoid-levels. This approximation of the superficies between a function and the x-axis is called trapezoid-rule. This rule only applies for constant and differentiaable functions which are limited to the interval [a,b]. If one subdivides this interval into many small and regular subintervals with the shape of a trapezoid and after this one sums up all of the trapezoid-levels, the outcome will be the trapezoid-procedure.

$$\int_a^b f_n(x)\,dx=\frac{b-a}{n}\,[\frac{1}{2}\,f(a)+f(a_1)+\ldots+f(a_{n-1})+\frac{1}{2}\,f(b)]$$

4 Keplersche-Faß-rule and Simpsonrule

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A short biography of Johannes Kepler (1571-1630)

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Johannes Kepler

The German theologian, astronomer and mathematician Johannes Kepler was born on 27.12.1571 in the town Weil, today Württemberg. After a theology-study of about ten years he moved to Graz, where a position as a mathematics-professor was offered to him. As a consequence of the counter-reformation Kepler had o flee to Prag in 1600. There he entered in the service off the emperor Rudolph Π as astronomer at court and mathematician. In and after 1628 Kepler mainly worked for the sovereign Wallenstein in Sagan. Two years later Kepler died unexpected during a visit of the "Kurfürstentage" in Regensburg on 15.11.1630.

Kepler got renowned for many of his works, mainly concerning astronomy and optics. Among them there are the “Grundgesetze der Planetenbewegung”, the first and second of the keplersche laws, which can be found in “Astronomia Nova”, and the third of the keplersche laws in “Harmonices Mundi”. Further there is an invention of a astronomical telescope and the “Rudolphinischen Tafeln”.
In one of his works, “nova stereometria doliorum vinariorum", Kepler used heuristically and infinitesimaltly methods, which are also employed in the fass-rule to calculate the superficies.

Keplersche Fass-rule

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If one knows the function-values of three equidistantlly supporting spots x_0, x_0+h, x_0+2h the following formula can be used:

$$\int_a^b f_1(x)\,dx = \frac{b-a}{6}[f(a)+4f(a_1)+f(b)]$$

Derivation of the Keplerschen Fass-rule

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One looks for the level F which limites the quadratical function $f(x)=b_2x^2+b_1x+b_0$ by the interval $[a_0;a_2]$ and the x-acis.

$$F \equiv \int_{a_0}^{a_2} f(u)\,du$$

One is given the function value by the points a_0, a_1, a_2, this means:

$$b_2a_0^2+b_1a_0+b_0=f(a_0)$$
$$b_2a_1^2+b_1a_1+b_0=f(a_1)$$
$$b_2a_2^2+b_1a_2+b_0=f(a_2)$$

$$\text{mit } a_1=\frac{a_0+a_2}{2}\,*.$$

$$F=\int_{a_0}^{a_2} f(u)\,du=\int_{a_0}^{a_2}(b_2u^2+b_1u+b_0)\,du$$
$$=b_2\frac{1}{3}u^3+b_1\frac{1}{2}u^2+b_0u\Big|_{a_0}^{a_2}$$
$$=b_2\frac{1}{3}a_2^3+b_1\frac{1}{2}a_2^2+b_0a_2-(b_2\frac{1}{3}a_0^3+b_1\frac{1}{2}a_0^2+b_0a_0)$$
$$=b_2\frac{1}{3}(a_2^3-a_0^3)+b_1\frac{1}{2}(a_2^2-a_0^2)+b_0(a_2-a_0)$$
$$=b_2\frac{1}{3}(a_2-a_0)(a_2^2+a_2a_0+a_0^2)+b_1\frac{1}{2}(a_2-a_0)(a_2+a_0)+b_0(a_2-a_0)$$
$$=\frac{a_2-a_0}{6}[2b_2(a_2^2+a_2a_0+a_0^2)+3b_1(a_2+a_0)+6b_0]$$
$$*=\frac{a_2-a_0}{6}[b_2(a_2^2+4\left(\frac{a_0-a_2}{2}\right)^2+a_0^2)+6b_1a_1+6b_0]$$
$$*=\frac{a_2-a_0}{6}[b_2(a_2^2+4a_1^2+a_0^2)+b_1(a_2+4a_1+a_0)+b_0(1+4+1)]$$
$$=\frac{a_2-a_0}{6}[b_2a_0^2+b_1a_0+b_0+4(b_2a_1^2+b_1a_1+b_0)+b_2a_2^2+b_1a_2+b_0]$$
$$=\frac{a_2-a_0}{6}(f(a_0+4f(a_1)+f(a_2))$$

The keplersche fassrule can be used for completerationally functions up to a degree of three. If it is used for other functions, the rule has to be applied more often. To avoid this expense the simpson rule is used as an extension of the kelp. faßrule.

Simpson - rule

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This is the simpson-rule:

$$\int_a^b f_m(x)\,dx=\frac{b-a}{3m}[f(a)+4f(a_1)+2f(a_2)+\dots+4f(a_{m-3})+2f(a_{m-2})+4f(a_{m-1})+f(b)]$$

5

Calculated fuctions - different integration - methods

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Task

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The exercises used as the basis of this paper come from some of the differen types of tasks. The functions used are: an expoential-, a root-, a rezipoke-, a Sin-, a Cos-, a Ln-, and an potenz-function. The single taskes are:

1.) Sin(x) $[0;\frac{\pi}{2}]$
2.) x^2 [0;5]
3.) Cos(x) $[-\frac{\pi}{2};0]$
4.) e^{x-9} [-1;2,5]
5.) $\sqrt{x+9}$ [0;1000]
6.) $\frac{1}{x+3}$ [0;15]
7.) Ln(x) [1;5]

The example 5.) to 7.) are not shown.

```
        Clear[f, x];     f[x_] := Sin[x] ; (* enter your f *)
        a = 0;
        b = π/2;
        start = 0;
        stop = 10;
Input ▷ step = 1;
```

```
tablevalues = Table[{n, ∑_{i=0}^{n-1} f[a + i (b - a)/n] * (b - a)/n // N, ∑_{i=1}^{n} f[a + i (b - a)/n] * (b - a)/n // N},
   {n, start, stop, step}];
MDShowTable[tablevalues, {"n", "Untersumme", "Obersumme"}]
```

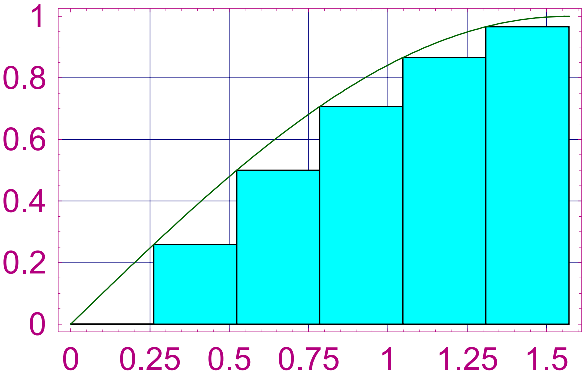
n	Untersumme	Obersumme
0	0.	0.
1	0.	1.5708
2	0.55536	1.34076
3	0.715249	1.23885
4	0.790766	1.18347
5	0.834682	1.14884
6	0.863382	1.12518
7	0.8836	1.108
8	0.89861	1.09496
9	0.910194	1.08473
10	0.919403	1.07648

```
Clear[f, x];     f[x_] := Sin[x]
a = 0;
```

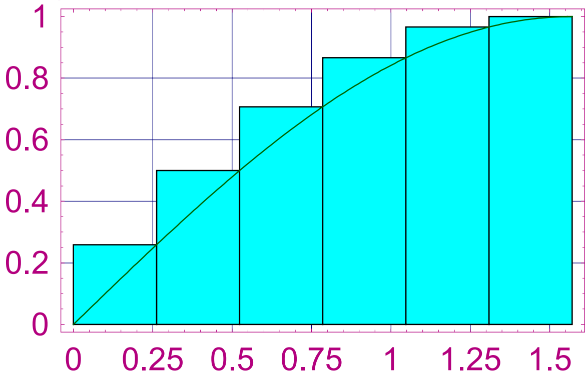
```
b = π/2;
start = 1; (* rectangles *)
end = 15;

MDIRiemannSum[f[x], {x, a, b},
    MovieFrames → {start, end}, RiemannSum → {Left, Right}]
```

n = 0006, Area = 1.
Left−hand Sum = +0.863382194392583



n = 0006, Area = 1.
Right−hand Sum = +1.12518158219173



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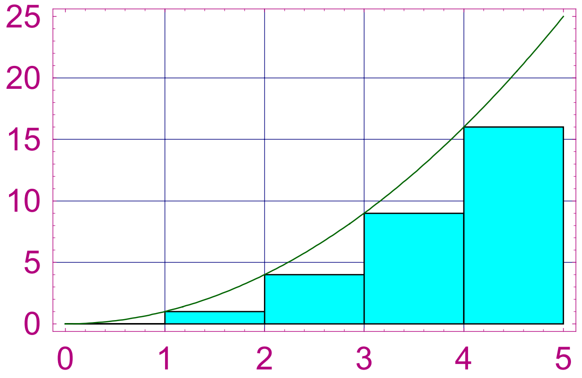
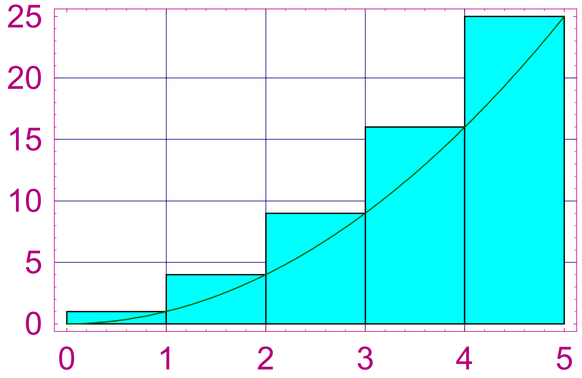
```
         Clear[f , x];
         f[x_] := x^2
         a = 0;
         b = 5;
Input ▷  start = 1; (* rectangles *)
         end = 10;

         MDIRiemannSum[f[x], {x, a, b},
             MovieFrames → {start, end}, RiemannSum → {Left, Right}]
```

n = 0005, Area = 41.66666666666667
Left−hand Sum = +000000000000030.



n = 0005, Area = 41.66666666666667
Right−hand Sum = +000000000000055.



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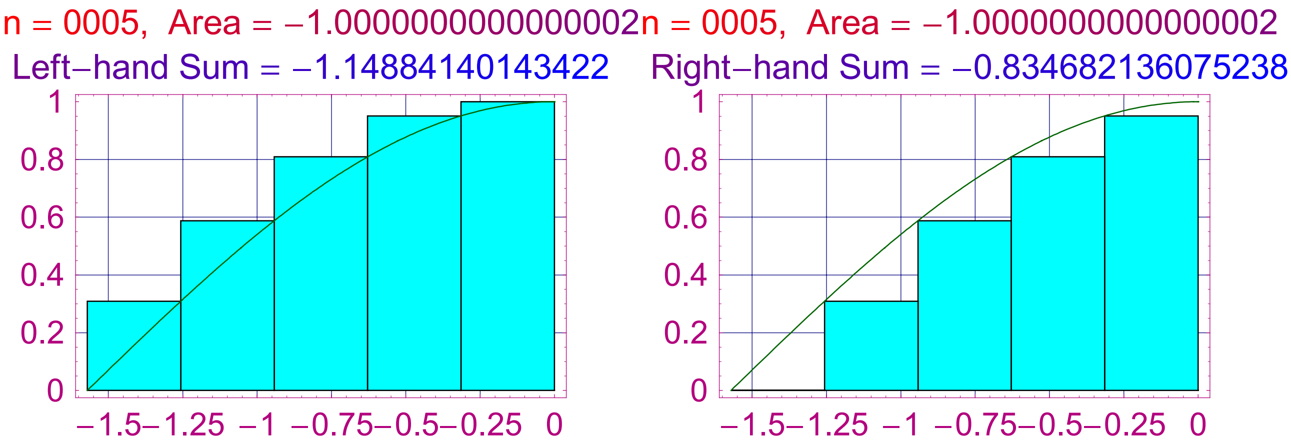
```
Clear[f , x];
f[x_] := Cos[x]
a = -π/2;
```

```
b = 0;
start = 1; (* rectangles *)
end = 10;

MDIRiemannSum[f[x], {x, a, b},
    MovieFrames → {start, end}, RiemannSum → {Left, Right}]
```



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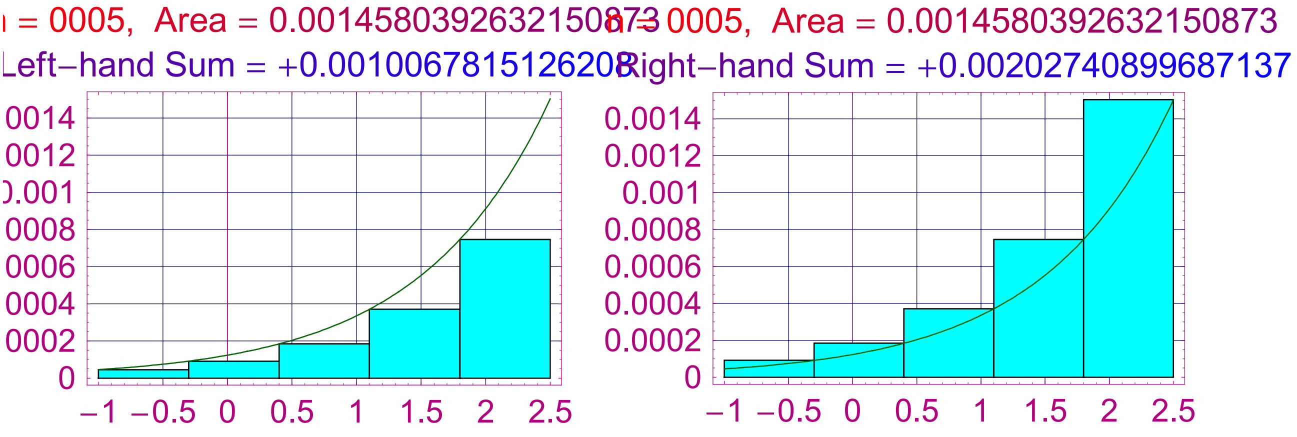
Input ▹

```
Clear[f , x];
f[x_] := ⅇ^(x-9)
a = -1;
b = 2.5;
start = 1; (* rectangles *)
end = 19;

MDIRiemannSum[f[x], {x, a, b},
    MovieFrames → {start, end}, RiemannSum → {Left, Right}]
```



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6 Self-developed circle-procedure

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Contents and purpose of the task

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For the determination of the surface between the curve, the x-acis and two boundaries [a,b], a<b, one has to develope a approximation procedure, which calculates the level by using circles. The steps get smaller and smaller (n→ n+1) and thus the approximate value of the integral of this curve-paragraph is developed. In order to be able to aplly this procedure, one needs a constantly differentiable function in the form y=f(x).

Solution of the task

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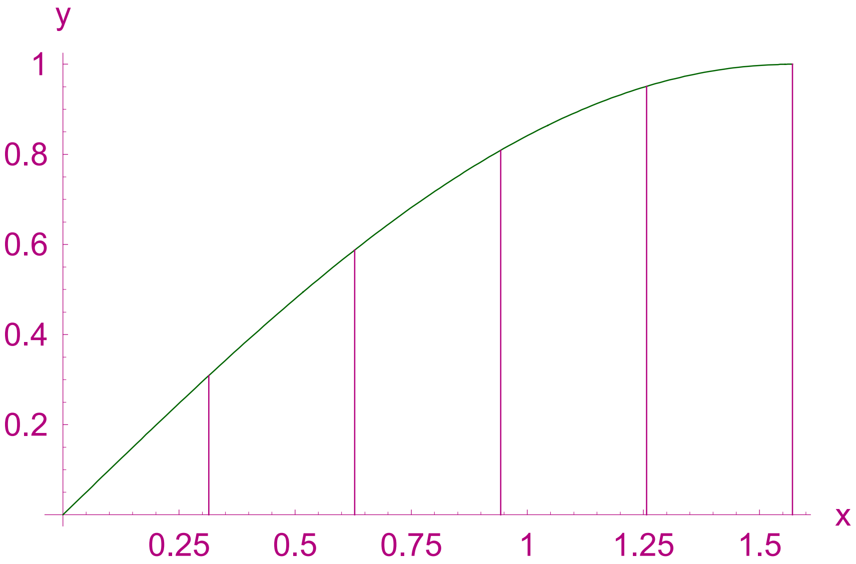
<u>Example:</u> The interval is divided in n(=5) equivalnet vertical strips. The sets, which "fill" the surface that can be determined , have the diameter 1/5 von Pi/2. The focal points are on straight lines between the boundaries as vertical strips. Sets, which are covered completely by the curve, are fully counted. Partly covered sets only considere, if the curve cuts the vertical set diameter which is situated paralell to the y-axis, however only with the surface below the function is situated. Therefore the second set of circles in Fig. 1 would be evaluated e.g. with approx. 1,5 sets. More exactly is the example : $x=\frac{9}{10}$ von $\frac{\pi}{2}$, where one comes to the result, that 3,144 circle fit into the surface.

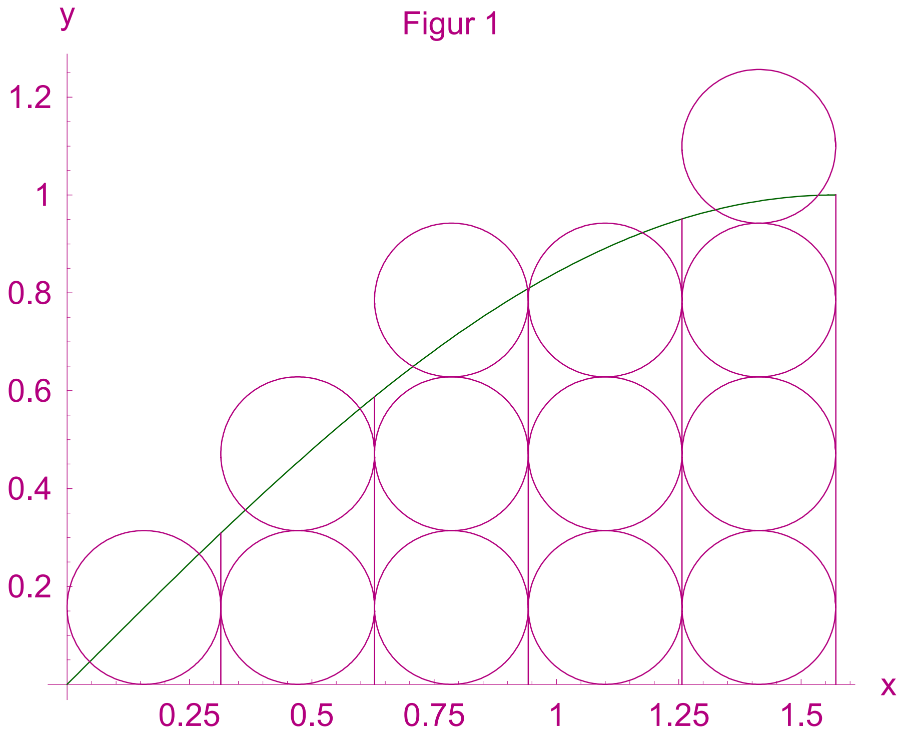
Input ▷

```
Clear[f, x];
f[x_] := Sin[x]
Graph = Plot[f[x], {x, 0, π/2}, AxesLabel -> {"x", "y"}, Epilog -> {
    Line[{{π/10, 0}, {π/10, f[π/10]}}],
    Line[{{π/5, 0}, {π/5, f[π/5]}}],
    Line[{{3 π/10, 0}, {3 π/10, f[3 π/10]}}],
    Line[{{2 π/5, 0}, {2 π/5, f[2 π/5]}}],
    Line[{{π/2, 0}, {π/2, f[π/2]}}]}];
```



Input ▹

```
r = π/20;
Kreise = Show[Graphics[
      {Circle[{π/20, π/20}, r]
      , Circle[{π/20 + 2 r, π/20}, r]
      , Circle[{π/20 + 4 r, π/20}, r]
      , Circle[{π/20 + 6 r, π/20}, r]
      , Circle[{π/20 + 8 r, π/20}, r]
      , Circle[{π/20 + 2 r, π/20 + 2 r}, r]
      , Circle[{π/20 + 4 r, π/20 + 2 r}, r]
      , Circle[{π/20 + 6 r, π/20 + 2 r}, r]
      , Circle[{π/20 + 8 r, π/20 + 2 r}, r]
      , Circle[{π/20 + 4 r, π/20 + 4 r}, r]
      , Circle[{π/20 + 6 r, π/20 + 4 r}, r]
      , Circle[{π/20 + 8 r, π/20 + 4 r}, r]
      , Circle[{π/20 + 8 r, π/20 + 6 r}, r]}, DisplayFunction → Identity]];
Show[Graph, Kreise, AspectRatio -> 0.8, PlotLabel -> "Figur 1"];
```



Such a procedure is permitted, since with sufficiently large n the sets can be made arbitrarily small: For a high number of n the sets become points and the limit value bemoces the integral!

n = 1 The mitdpoint is situated in x = Pi/4
that means, y = Sin Pi/4 = Sin 45° = 0,7071
the circle diameter is Pi/4 = 0,7854

$$\frac{0,\,7071}{0,\,7854}=0{,}9$$

For n = 5:

Input ▷

```
Clear[f, x];     f[x_] := Sin[x] ; (* enter your f *)
start = 1;
stop = 11;
step = 2;

tablevalues = Table[{πx/20, f[πx/20], 10 f[πx/20]/π} // N, {x, start, stop, step}];
MDShowTable[tablevalues, {"Bogenmaß", f[x], "Kreise"}];
```

Bogenmaß	Sin[x]	Kreise
0.15708	0.156434	0.497946
0.471239	0.45399	1.4451
0.785398	0.707107	2.25079
1.09956	0.891007	2.83616
1.41372	0.987688	3.14391
1.72788	0.987688	3.14391

A circle which contains the diamtere of 0,31416 also has the surface of:

$$F=\frac{D^2}{4}\times\pi=0{,}0775$$

Basic numberl (circle) = Sum of circle parts * the surface of the circles
=13,1329* 0,0775 =1,0178

The now known value of n = 6 is 1,0178, the real value of the integral is 1,0.

After 6 steps, the misstake is:

$$\frac{1,0-0,0178}{1} = 1,78\%$$

7

Evaluation

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General analysis

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Comprising all the data presented one can say that the procedure, which calculates for n=1 the closest outcome of the value of the integral, also delivers for n=15 the most exact value. This statement can be applied to the simpson-rule. On the other side there is the approximation of the Untersumme, which always results in the most inaccurate outcome. The rectangle-rule, trapezoid-rule and the Obersumme follow right after the simpson-rule. It ensues out of this order of precedence that the simpson-rule is 22 time more accurate as the trapezoid-rule, ca.13 times more exact as the rectangle-procedure, ca. 757 times more accurate as the Obersumme and ca. 809 times more accurate as the Untersumme.

The average mistake of the Untersumme totals to 0,0947852, the average of the Obersumme to 0,0886888. The mistake of procedure of the rectangle-rule amounts 0,00158 which strives for $n \to \infty$ after zero, but it only shrinks proportional to n. Because of this, n has to be timed by ten to improve the precision by a decimal place. In most cases formula (3) is more accurate then formula (1) and (2) if n is given. In every approximation-procedure the accuracy of all formulas increases as the number of n increases. The trapezoid-rule accounts a value of 0,002546 which decreases proportionally to n^2. Last but not least there is the simpson-rule which calculates the top value of 0,0001171.

Speed of approximation (A)

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If one wants to calculate the velocity of approximation, he has to compute the needed steps -n- to achieve the required velocity -G-. This one can do by using the following formula:

$$A=\frac{n}{G},$$

but in the course of this, one has to proceed from the worst result to estimate the velocity, because otherwise the steps needed to do so by e.g. the Untersumme could get too high.

The chosen velocity equals 8%, which is achieved after the second step by the self-developed circle-procedure. 8% of the ln(x)-function are equal to 0,0566066. It will not be looked at the simpson-rule, because it already starts with a value of 0,044599. After the second step, the trapezoid- and the rectangle-rule have subdued the 8% hurdle. At the point of n=10 and n=11 also the Ober- und Untersumme reach the chosen velocity.

If one looks at the rezipoke-function it all seems to be different. After the first, second and third step the demanded velocity is obtained. E-function, root-function, Sin(x), Cos(x)can be each added to one of the two possibilities of velocity approximation.

The root-function, Sin(x) and Cos(x) look a lot like the Ln(x)-function: after the second steps trapezoid- and rectangle rule have reached the 8%, the simpson-procedure cannot be counted because of its accurate values, and the Ober- und Untersumme have reached the velocity after the tenth and elefenth step.

The exponential-function doesn’t cut off as good as the Ln(x)-function, but it accounts the velocity after the second, third and fourth step. Therefore this procedure is more accurate as the rezipoke-function.

Last there is the ponential-function, forming a special group of its own, because trapezoid-, simpson-, and rectangle-rule all have the same value. This value is already at the point of n=1 as accurate as by n=10.000. Consequently it does not change. As these rules established a far to good result, the Ober- und Untersumme are obmitted way to bad. The velocity of 8% is reached at as a high number of n as 14.

General speaking, one can say that the trapezoid-, simpson-, and rectangle-rule cut off as the best, but most of the time one cannot pay attention to the simpson-rule, because the value at the beginning with n=1 surmounts the velocity right from the beginning. After a high number of n is used the Ober- and Untersumme achieve the wanted velocity.

Summary

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After an solid evaluation of the single approximation-procedures, applied to different functions, one can say that the simpson-rule accounts the most accurate results. To be close to the real value of the integral is desirable, but one has to pay attention to the expense used to achieve the value. In cybernetic, by steering-procedures, in chemistry, for the supervision of single processes, it doesn't really matter if the calculated result is four or fife percent to far away from the velocity, the time used to achieve this value is more important. If one uses the Obersumme he will get to a result by dividing the length of the interval, b-a, into n-steps. After this he multiplies the result by the right function value. If one looks at the Simpson-rule, he will see that it delivers a 757 times more accurate result, but it also needs three known equidistantly supporting spots and this procedure is far more complicated compared to the Obersumme.

Therefore one integration-method can be useful for one thing, but because of its expense not be useful for other things. Fortunately we are living in the 21th century, the time of the internet and computers, which can calculate in a few seconds every integration-method. Consequently the reaction-time in e.g. air plains is not as long as it was 5 years ago.

Final statement

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As many years as the numerical integration exists, there are as many different integration-methods. In this paper only a few of so many procedures were used. Anyways, I think that the usage of a few but well known approximation-methods for a comparison are good enough, to provide a overview.

8

Source material

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General sources

Öffnen / Schließen

1. Leistungskurs Analysis
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Ernst Klett Verlag, 2002
Stuttgart Düsseldorf Leipszig
2. dtv-Atlas, Analysis und angewandte Mathematik
Band 1 und 2
Deutscher Taschenbuch Verlag, 2001
3. Taschenbuch der Mathematik
Von Bronstein, Semendjajew, Musiol, Mühlig
Verlag Harri Deutsch, 5. Auflage 2001
4. Höhere Mathematik griffbereit
Von M.J.Wygodski
Friedr. Vieweg & Sohn-Braunschweig, 2. Auflage 1977

Kepler sources

Öffnen / Schließen

1. Mathematiker-Lexikon
   Von Herbert Meschkowski
   Hochschultaschenbücher Bibliographische Institut, 1964
   Seite 138-139
2. Biographien bedeutender Mathematiker
   Von Hans Wußing und Wolfgang Arnold
   Aulis Verlag Deubner & Co Köln, 1978
   Seite 132-148
3. Lexikon der Schulmathematiker
   Von H. Athen und J. Bruhn
   Aulis Verlag Deubner & Co KG, 1977
   Seite 467
4. abc Mathematik
   von Walter Gellert und Herbert Kästner und Dr. Siegfrie Neuber
   Harri Deutsch, 1978
5. picture of Kepler:
   www.geocities.com/CapeCanaveral/Hangar/6580/webdoc1.htm

Input ▷

```
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        Background → RGBColor[0.808606, 0.921889, 0.8047]],
       FontSlant → "Plain", FontColor → RGBColor[0.062501, 0.644541, 0.062501]],
      RowBox[{" ", StyleBox[ButtonBox[AdjustmentBox[RowBox[
            {"  ", RowBox[{RowBox[{"Kapitel", " ", "ausschneiden"}], " "}]}],
           BoxMargins → {{0, 0}, {0, 0.4}}],
          ButtonData :→ "Section", ButtonStyle → "MDCutGroup",
          Background → RGBColor[0.808606, 0.921889, 0.8047]], FontSlant → "Plain",
         FontColor → RGBColor[0.062501, 0.644541, 0.062501]], "     "}]}},
    ColumnSpacings → 0.5, GridFrame → {{0, 0}, {0, 0.25}},
    GridFrameMargins →
     {{0.4, 0.4}, {0.5, 1}}],
   FontSize → 12, FontColor →
    RGBColor[0.665293,
     0.0723888, 0.101137],
   ButtonBoxOptions → {ButtonMargins → 1}]],
  "Text", ShowCellBracket →
   False,
  CellMargins →
   {{Inherited, 0},
    {Inherited, 55}},
  CellElementSpacings →
   {"ClosedGroupTopMargin" → 55},
  CellGroupingRules →
   {"SectionGrouping",
    15},
```

```
TextAlignment → Center,
TextJustification →
 0,
CellTags →
 "MDNoHTML"]
```

Neues Kapitel Kapitel ausschneiden

Drucken Öffnen / Schließen

Erzeuge deinen eigenen surfing GUIDE für Mathematik!